

Flavoured large N gauge theory in an external magnetic field

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ABSTRACT: We consider a D7-brane probe of $\text{AdS}_5 \times S^5$ in the presence of pure gauge B -field. In the dual gauge theory, the B -field couples to the fundamental matter introduced by the D7-brane and acts as an external magnetic field. The B -field supports a 6-form Ramond-Ramond potential on the D7-branes world volume that breaks the supersymmetry and enables the dual gauge theory to develop a non-zero fermionic condensate. We explore the dependence of the fermionic condensate on the bare quark mass m_q and show that at zero bare quark mass a chiral symmetry is spontaneously broken. A study of the meson spectrum reveals a coupling between the vector and scalar modes, and in the limit of weak magnetic field we observe Zeeman splitting of the states. We also observe the characteristic $\sqrt{m_q}$ dependence of the ground state corresponding to the Goldstone boson of spontaneously broken chiral symmetry.

KEYWORDS: AdS-CFT Correspondence, D-branes.

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1. Introduction

In recent years, progress has been made in the study of gauge theory with matter in the fundamental representation in the context of gauge/string dualities generalizing the AdS/CFT correspondence. One way to achieve this is by introducing D7-branes in the probe limit [2] that amounts to the condition $N_f \ll N_c$. The fundamental strings stretched between the stack of N_c D3-branes and the N_f flavor D7-branes give rise to $\mathcal{N}=2$ hypermultiplet. The separation of the D3- and D7-branes in the transverse directions corresponds to the mass of the hypermultiplet, the classical shape of the D7-brane encodes the value of the fermionic condensate, and its fluctuations describe the light meson spectrum of the theory [3]. This technique for introducing fundamental matter has been widely employed in different backgrounds. Of particular interest is the study of non supersymmetric backgrounds and phenomena such as spontaneous chiral symmetry breaking. These phenomena were first studied in this context in ref. [4], using analytical and numerical techniques. In several works this approach was further developed, and has proven itself a powerful tool for the exploration of gauge theories, in particular, for the description of their thermodynamic properties or for the building of phenomenological models relevant to QCD [5]–[40].

In this paper we will be interested in introducing fundamental matter into the gauge theory in the presence of an external electromagnetic field that couples to the fundamental fermions. The supersymmetry will be explicitly broken by the external field, and we will observe spontaneous symmetry breaking, and non-trivial mixing in the spectrum of mesons.

There have been non-perturbative studies of fermionic models in background magnetic field before, and there is a large literature (see e.g., the reviews of refs. [43, 44], and the

discussion of ref. [45] and references therein). Generally, those works use quite different methods to examine aspects of the physics — some primary non-perturbative tools are the Dyson-Schwinger equations in various truncations). Our results are consistent with the general expectations from those works, which is that strong magnetic fields are generically expected to be a catalyst for spontaneous chiral symmetry breaking in a wide class of models (see e.g., refs. [46, 45, 43] for a discussion of the conjectured universality of this result).

2. Fundamental matter in an external magnetic field

2.1 Basic configuration

Let us consider the $\text{AdS}_5 \times S^5$ geometry describing the near-horizon physics of a collection of N_c extremal D3-branes.

$$\begin{aligned} ds^2 &= \frac{u^2}{R^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{du^2}{u^2} + R^2 d\Omega_5^2, \\ g_s C_{(4)} &= \frac{u^4}{R^4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\ e^\Phi &= g_s, \\ R^4 &= 4\pi g_s N_c \alpha'^2, \end{aligned} \tag{2.1}$$

Where $d\Omega_5^2$ is the unit metric on a round S^5 . In order to introduce fundamental matter we first rewrite the metric in the following form, with $d\Omega_3^2$ the metric on a unit S^3 :

$$\begin{aligned} ds^2 &= \frac{\rho^2 + L^2}{R^2}[-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2] + \frac{R^2}{\rho^2 + L^2}[d\rho^2 + \rho^2 d\Omega_3^2 + dL^2 + L^2 d\phi^2], \\ d\Omega_3^2 &= d\psi^2 + \cos^2 \psi d\beta^2 + \sin^2 \psi d\gamma^2, \end{aligned} \tag{2.2}$$

where $\rho, \psi, \beta, \gamma$ and L, ϕ are polar coordinates in the transverse \mathbb{R}^4 and \mathbb{R}^2 respectively. Note that: $u^2 = \rho^2 + L^2$. We use $x_0, x_1, x_2, x_3, \rho, \psi, \beta, \gamma$ to parametrise the world volume of the D7-brane and consider the following ansatz [3] for its embedding:

$$\phi \equiv \text{const}, \quad L \equiv L(\rho),$$

leading to the following form of the induced metric on its worldvolume:

$$d\tilde{s} = \frac{\rho^2 + L(\rho)^2}{R^2}[-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2] + \frac{R^2}{\rho^2 + L(\rho)^2}[(1 + L'(\rho)^2)d\rho^2 + \rho^2 d\Omega_3^2]. \tag{2.3}$$

Now let us consider the general DBI action:

$$S_{\text{DBI}} = -\mu_7 \int_{\mathcal{M}_8} d^8 \xi e^{-\Phi} [-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})]^{1/2}. \tag{2.4}$$

Here $\mu_7 = [(2\pi)^7 \alpha'^4]^{-1}$ is the D7-brane tension, G_{ab} and B_{ab} are the induced metric and B -field on the D7-brane's world volume, while F_{ab} is its world-volume gauge field. A

simple way to introduce magnetic field would be to consider a pure gauge B -field along parts of the D3-branes' world volume, e.g.:

$$B^{(2)} = H dx_2 \wedge dx_3 . \tag{2.5}$$

Since B_{ab} can be mixed with the gauge field strength F_{ab} , this is equivalent to a magnetic field on the world-volume. Recently a similar approach was used to study drag force in SYM plasma [36]. Note that since the B -field is pure gauge, $dB = 0$, the corresponding background is still a solution to the supergravity equations of motion. On the other hand, the gauge field F_{ab} comes at next order in the α' expansion compared to the metric and the B -field components. Therefore to study the classical embedding of the D-brane one can study only the $(G_{ab} + B_{ab})$ part of the DBI-action. However, because of the presence of the B -field, there will be terms of first order in α' in the full action linear in the gauge field F_{ab} . Hence integrating out F_{ab} will result in a constraint for the classical embedding of the D7-brane.

Since for our configuration, we have that:

$$B^{(2)} \wedge B^{(2)} = 0, \quad B^{(2)} \wedge C_{(4)} = 0,$$

and at first order in α' the only contribution to the Wess-Zummino is

$$2\pi\alpha'\mu_7 \int F_{(2)} \wedge C_{(6)} . \tag{2.6}$$

By using the following expansion in the DBI action:

$$[-\det(E_{ab} + 2\pi\alpha'F_{ab})]^{1/2} = \sqrt{E} + \pi\alpha'\sqrt{E}E^{ba}F_{ab} + O(F^2); \quad E = -\det E_{ab}; , \tag{2.7}$$

where we have introduced $E_{ab} = G_{ab} + B_{ab}$ as a notation for the generalized induced metric, we obtain the following action to first order in α' :

$$S_F = \pi\alpha' \frac{\mu_7}{g_s} \int_{\mathcal{M}_8} d^8\xi \sqrt{E} E^{[ab]} F_{[ab]} + 2\pi\alpha'\mu_7 \int F_{(2)} \wedge C_{(6)} . \tag{2.8}$$

Note that in equation (2.8) we didn't take the $g_s \rightarrow 0$ limit, which would suppress the Wess-Zummino term, provided that the $C_{(6)}$ potential had no explicit g_s dependence. The resulting equation of motion does not contain A_a and sets the following constraint for the $C_{(6)}$ potential induced by the gauge B -field.

$$\frac{g_s}{6!} \epsilon^{ab\tilde{\mu}_1 \dots \tilde{\mu}_6} \partial_a C_{\tilde{\mu}_1 \dots \tilde{\mu}_6} = -\partial_a (\sqrt{E} E^{[ba]}); \quad a, b, \tilde{\mu}_1, \dots, \tilde{\mu}_6 \in \mathcal{M}_8; . \tag{2.9}$$

Note that $C_{(6)}$ has a dynamical term proportional to $1/\kappa_0^2$ in the supergravity action, and that the D7-brane action is proportional to $\mu_7 = 2\pi/\kappa_0^2$. Therefore they are at the same order in α' and g_s [42]. We must solve for $C_{(6)}$ using the action:

$$S_{C_{(6)}} = \mu_7 \int B_{(2)} \wedge C_{(6)} - \frac{1}{4k_0^2} \int d^{10}x \sqrt{-G} |dC_{(6)}|^2 . \tag{2.10}$$

The solution obtained from equation (2.10) has to satisfy the constraint given in equation (2.9). Our next goal will be to find a consistent ansatz for $C_{(6)}$. To do this let us consider the classical contribution to the DBI action:

$$S_{\text{NS}} = -\frac{\mu_7}{g_s} \int d^8\xi \sqrt{E} . \quad (2.11)$$

From equation (2.11) one can solve for the classical embedding of the D7-brane, which amounts to second order differential equation for $L(\rho)$ with some appropriate solution $L_0(\rho)$. After substituting $L_0(\rho)$ in (2.11) we can extract the form of the $C_{(6)}$ potential induced by the B -field. However one still has to satisfy the constraint (2.9). It can be verified that with the choice (2.5) for the B -field and the ansatz of equation (2.3) for the induced metric, the right-hand side of equation (2.9) is zero. Then equation (2.9) and the effective action (2.10) boil down to finding a consistent ansatz for $C_{(6)}$ satisfying:

$$\partial_\mu(\sqrt{-G}dC_6^{\mu 01\rho\psi\alpha\beta}) = -\frac{\mu_7\kappa_0^2}{\pi}H\delta(L - L_0(\rho)) , \quad (2.12)$$

$$\text{or} \quad \partial_\mu(\sqrt{-G}dC_6^{\mu 01L\psi\alpha\beta}) = -L'_0(\rho)\frac{\mu_7\kappa_0^2}{\pi}H\delta(L - L_0(\rho)) , \quad (2.13)$$

$$\epsilon^{ab\tilde{\mu}_1\dots\tilde{\mu}_6}\partial_a C_{\tilde{\mu}_1\dots\tilde{\mu}_6} = 0; \quad a, b, \tilde{\mu}_1, \dots, \tilde{\mu}_6 \in \mathcal{M}_8; . \quad (2.14)$$

One can verify that the choice:

$$C_{(6)01\rho\psi\alpha\beta} = \frac{1}{7}f(\rho, L, \psi), \quad dC_{(6)L01\rho\psi\alpha\beta} = \partial_L f , \quad (2.15)$$

is a consistent ansatz and the solution for the $C_{(6)}$ field strength can be found to be:

$$dC_{(6)L01\rho\psi\alpha\beta} = \frac{\mu_7\kappa_0^2}{\pi}H\frac{\rho^3R^4}{L(\rho^2 + L^2)^2}\Theta(L - L_0(\rho))\sin\psi\cos\psi . \quad (2.16)$$

It is this potential which breaks the supersymmetry. Note that any solution for the $C_{(6)}$ potential satisfying equations (2.12) and (2.13) would not depend on g_s , therefore in the $g_s \rightarrow 0$ limit we could suppress the Wess-Zumino term in equation (2.8) and relax the additional constraint (2.14). It is important to note that there is no contradiction between the fact that the B -field that we have chosen does not break the supersymmetry of the $\text{AdS}_5 \times S^5$ supergravity background, on the one hand, and the fact that the physics of the D7-brane probing that background does have supersymmetry broken by the B -field, on the other. This is because the physics of the probe does not back-react on the geometry.

In what follows, we will study the physics of the D7-branes and the resulting dual gauge theory physics. Among the solutions for the D7-brane embedding, there will be a class with non-trivial profile having zero asymptotic separation between the D3- and D7-branes. This corresponds to a non-zero fermionic condensate at zero bare quark mass. Therefore the non-zero background magnetic field will spontaneously break the chiral symmetry. Geometrically this corresponds to breaking of the $\text{SO}(2)$ rotational symmetry in the (L, ϕ) -plane [3].

2.2 Properties of the solution

We now proceed with the exploration of the properties of the classical D7-brane embedding. If we consider the action (2.11) at leading order in α' , we get the following effective lagrangian:

$$\mathcal{L} = -\frac{\mu_7}{g_s} \rho^3 \sin \psi \cos \psi \sqrt{1 + L'^2} \sqrt{1 + \frac{R^4 H^2}{(\rho^2 + L^2)^2}}. \quad (2.17)$$

The equation of motion for the profile $L_0(\rho)$ of the D7-brane is given by:

$$\partial_\rho \left(\rho^3 \frac{L'_0}{\sqrt{1 + L_0'^2}} \sqrt{1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}} \right) + \frac{\sqrt{1 + L_0'^2}}{\sqrt{1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}}} \frac{2\rho^3 L_0 R^4 H^2}{(\rho^2 + L_0^2)^3} = 0. \quad (2.18)$$

As expected for large $(L_0^2 + \rho^2) \rightarrow \infty$ or $H \rightarrow 0$, we get the equation for the pure $\text{AdS}_5 \times S^5$ background [2]:

$$\partial_\rho \left(\rho^3 \frac{L'_0}{\sqrt{1 + L_0'^2}} \right) = 0.$$

Therefore the solutions to equation (2.18) have the following behavior at infinity:

$$L_0(\rho) = m + \frac{c}{\rho^2} + \dots \quad (2.19)$$

where the parameters m (the asymptotic separation of the D7- and D3- branes) and c (the degree of bending of the D7-brane) are related to the bare quark mass $m_q = m/2\pi\alpha'$ and the fermionic condensate $\langle \bar{\psi}\psi \rangle \propto -c$ respectively [5]. As we shall see below, the presence of the external magnetic field and its effect on the dual SYM provide a non vanishing value for the fermionic condensate, furthermore the theory exhibits chiral symmetry breaking.

Now notice that H enters in (2.17) only through the combination $H^2 R^4$. The other natural scale is the asymptotic separation m . It turns out that different physical configurations can be studied in terms of the ratio $\tilde{m}^2 = m^2/(HR^2)$: Once the \tilde{m} dependence of our solutions are known, the m and H dependence follows. Indeed let us introduce dimensionless variables *via*:

$$\rho = R\sqrt{H}\tilde{\rho}, \quad L_0 = R\sqrt{H}\tilde{L}, \quad L'_0(\rho) = \tilde{L}'(\tilde{\rho}). \quad (2.20)$$

The equation of motion (2.18) then takes the form:

$$\partial_{\tilde{\rho}} \left(\tilde{\rho}^3 \frac{\tilde{L}'}{\sqrt{1 + \tilde{L}'^2}} \sqrt{1 + \frac{1}{(\tilde{\rho}^2 + \tilde{L}^2)^2}} \right) + \frac{\sqrt{1 + \tilde{L}'^2}}{\sqrt{1 + \frac{1}{(\tilde{\rho}^2 + \tilde{L}^2)^2}}} \frac{2\tilde{\rho}^3 \tilde{L}}{(\tilde{\rho}^2 + \tilde{L}^2)^3} = 0 \quad (2.21)$$

The solutions for $\tilde{L}(\tilde{\rho})$ can be expanded again to:

$$\tilde{L}(\tilde{\rho}) = \tilde{m} + \frac{\tilde{c}}{\tilde{\rho}^2} + \dots, \quad (2.22)$$

and using the transformation (2.20) we can get:

$$c = \tilde{c} R^3 H^{3/2}. \quad (2.23)$$

It is instructive to study first the properties of (2.21) for $\tilde{m} \gg 1$, which corresponds to weak magnetic field $H \ll m^2/R^2$, or equivalently large quark mass $m \gg R\sqrt{H}$.

2.2.1 Weak magnetic field

In order to analyze the case of weak magnetic field let us expand $\tilde{L}(\tilde{\rho}) = \tilde{m} + \eta(\tilde{\rho})$ and linearize equation (2.21) while leaving only the leading terms in $(\tilde{\rho}^2 + \tilde{m}^2)^{-1}$. The result is:

$$\partial_{\tilde{\rho}} (\tilde{\rho}^3 \eta') + \frac{2\tilde{\rho}^3 \tilde{m}}{(\tilde{\rho}^2 + \tilde{m}^2)^3} = 0, \quad (2.24)$$

which has the general solution:

$$\eta(\tilde{\rho}) = \frac{C_1}{\tilde{\rho}^2} - \frac{\tilde{m}}{4\tilde{\rho}^2(\tilde{m}^2 + \tilde{\rho}^2)} + C_2. \quad (2.25)$$

From the definition of $\eta(\tilde{\rho})$ and equation (2.22) we can see that $C_1 = \tilde{c}$ and since $\eta|_{\tilde{\rho} \rightarrow \infty} = 0$ we have $C_2 = 0$. Now if we consider \tilde{m} large enough, equation (2.25) should be valid for all $\tilde{\rho}$. It turns out that if we require that our solution be finite as $\tilde{\rho} \rightarrow 0$ we can determine the large \tilde{m} behavior of \tilde{c} . Indeed the second term in (2.25) has the expansion:

$$-\frac{\tilde{m}}{4\tilde{\rho}^2(\tilde{m}^2 + \tilde{\rho}^2)} = -\frac{1}{4\tilde{m}} \frac{1}{\tilde{\rho}^2} + \frac{1}{4\tilde{m}^3} + O(\tilde{\rho}^2). \quad (2.26)$$

Therefore we deduce that:

$$C_1 = \tilde{c} = \frac{1}{4\tilde{m}}, \quad (2.27)$$

and finally, we get for the profile of the D7-brane for $\tilde{m} \gg 1$:

$$\tilde{L}(\tilde{\rho}) = \tilde{m} + \frac{1}{4\tilde{m}} \frac{1}{\tilde{\rho}^2} - \frac{\tilde{m}}{4\tilde{\rho}^2(\tilde{m}^2 + \tilde{\rho}^2)}. \quad (2.28)$$

If we go back to dimensionful parameters we can see, using equations (2.23) and (2.27) that for weak magnetic field H the theory has developed a fermionic condensate:

$$\langle \bar{\psi} \psi \rangle \propto -c = -\frac{R^4}{4m} H^2. \quad (2.29)$$

However this formula is valid only for sufficiently large m and we cannot make any prediction for the value of the fermionic condensate at zero quark mass. To go further, the involved form of equation (2.21) suggests the use of numerical techniques.

2.2.2 Numerical results

In this subsection we solve numerically equation (2.21) for the embedding of the D7-brane, using Mathematica. It is convenient to use initial conditions in the IR as has been recently discussed in the literature [6, 7]. We use the boundary condition $\tilde{L}'(\tilde{\rho})|_{\tilde{\rho}=0} = 0$. We used shooting techniques to generate the embedding of the D7 for a wide range of \tilde{m} . Having done so we expanded numerically the solutions for $\tilde{L}(\tilde{\rho})$ as in equation (2.22) and generated the points in the $(\tilde{m}, -\tilde{c})$ plane corresponding to the solutions. The resulting plot is presented in figure 1.

As one can see there is a non zero fermionic condensate for zero bare quark mass, the corresponding value of the condensate is $\tilde{c}_{\text{cr}} = 0.226$. It is also evident that the analytical

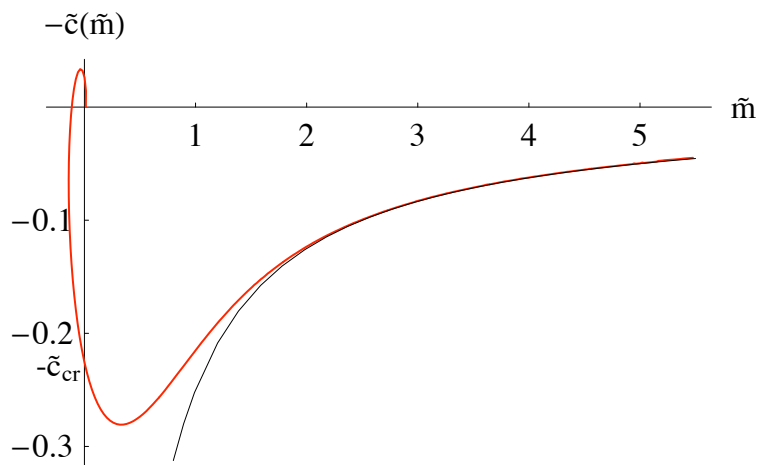


Figure 1: The black line corresponds to (2.27), one can observe that the analytic result is valid for large \tilde{m} . It is also evident that for $\tilde{m} = 0 \langle \bar{\psi}\psi \rangle \neq 0$. The corresponding value of the condensate is $\tilde{c}_{\text{cr}} = 0.226$

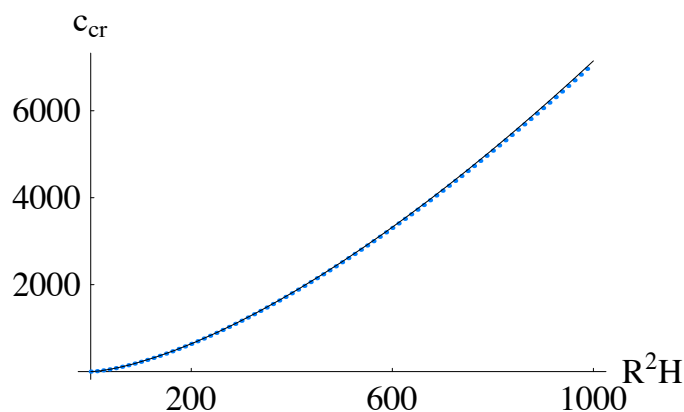


Figure 2: A plot of the magnitude of the fermionic condensate at zero bare quark mass c_{cr} as function of R^2H , the black curve represents equation (2.30).

expression for the condensate (2.27) that we got in the previous section is valid for large \tilde{m} , as expected. Now using equation (2.23) we can deduce the dependence of c_{cr} on H :

$$c_{\text{cr}} = \tilde{c}_{\text{cr}} R^3 H^{3/2} = 0.226 R^3 H^{3/2} . \tag{2.30}$$

It is interesting to check the consistency of our numerical analysis by solving equation (2.18) numerically and extracting the value of c_{cr} for wide range of R^2H , the resulting plot fitted with equation (2.30) is presented in figure 2.

Another interesting feature of our phase diagram is the spiral behavior near the origin of the $(\tilde{m}, -\tilde{c})$ -plane which can be seen in figure 3. A similar feature has been observed in ref. [7], where the authors have argued that only the lowest branch of the spiral correspond-

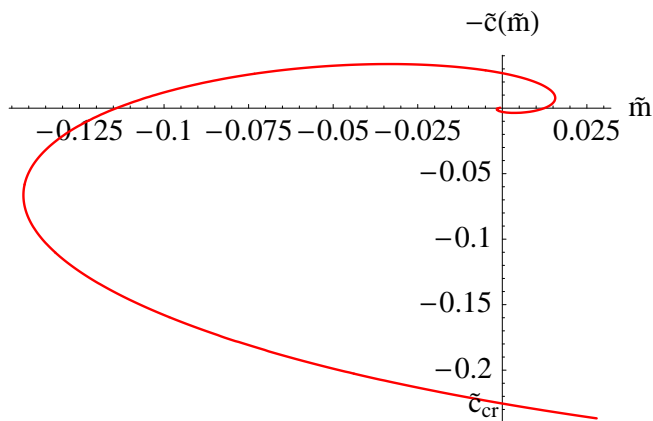


Figure 3: A magnification of figure 1 to show the spiral behavior near the origin of the $(-\tilde{c}, \tilde{m})$ -plane.

ing to positive values of m is the stable one (corresponding to the lowest energy state). The spiral behavior near the origin signals instability of the embedding corresponding to $L_0 \equiv 0$. If we trace the curve of the diagram in figure 3 starting from large m , as we go to smaller values of m we will reach zero bare quark mass for some large negative value of the fermionic condensate c_{cr} . Now if we continue tracing along the diagram one can verify numerically that all other points correspond to embeddings of the D7-brane which intersect the D3-brane at least once. (Note also that in ref. [4], such behavior was considered inconsistent with the interpretation of the embedding as a re-normalization group flow.) After further study one finds that the part of the diagram corresponding to negative values of \tilde{m} represents solutions for the D7-brane embedding which intersect the D3-branes odd number of times, while the positive part of the spiral represents solutions which intersect the D3-branes even number of times. The lowest positive branch corresponds to solutions which don't intersect the D3-branes and is the stable one, while the upper branches have correspondingly 2, 4, *etc.*, intersection points and are ruled out.

3. Meson spectrum

3.1 General properties

We study the scalar meson spectrum. To do so we will consider quadratic fluctuations [3] of the embedding of the D7-brane in the transverse (L, ϕ) -plane. It can be shown that because of the diagonal form of the metric the fluctuation modes along the ϕ coordinate decouple from the one along L . However, because of the non-commutativity introduced by the B -field we may expect the scalar fluctuations to couple to the vector fluctuations. This has been observed in ref. [9], where the authors considered the geometric dual to non-commutative super Yang Mills. In our case the mixing will be even stronger, because of the non-trivial profile for the D7-brane embedding, resulting from the broken supersymmetry.

Let's proceed with obtaining the action for the fluctuations. To obtain the contribution from the DBI part of the action we consider the expansion:

$$L = L_0(\rho) + 2\pi\alpha'\chi, \quad \phi = 0 + 2\pi\alpha', \quad (3.1)$$

where $L_0(\rho)$ is the classical embedding of the D7-brane solution to equation (2.18). To second order in α' we have the following expression:

$$E_{ab} = E_{ab}^0 + 2\pi\alpha'E_{ab}^1 + (2\pi\alpha')^2 E_{ab}^2, \quad (3.2)$$

where E^0, E^1, E^2 are given by:

$$\begin{aligned} E_{ab}^0 &= G_{ab}(\rho, L_0(\rho), \psi) + B_{ab}, \\ E_{ab}^1 &= \frac{R^2 L_0'}{\rho^2 + L_0^2} (\partial_a \chi \delta_b^\rho + \partial_b \chi \delta_a^\rho) + \partial_{L_0} G_{ab} \chi + F_{ab} \\ E_{ab}^2 &= \frac{R^2}{\rho^2 + L_0^2} (\partial_a \chi \partial_b \chi + L_0^2 \partial_a \Phi \partial_b \Phi) - \frac{2R^2 L_0 L_0'}{(\rho^2 + L_0^2)^2} (\partial_a \chi \delta_b^\rho + \partial_b \chi \delta_a^\rho) \chi + \frac{1}{2} \partial_{L_0}^2 G_{ab} \chi^2. \end{aligned} \quad (3.3)$$

Here G_{ab} and B_{ab} are the induced metric and B field on the D7-brane's world volume. Now we can substitute equation (3.3) into equation (2.11) and expand to second order in α' . It is convenient [9] to introduce the following matrices:

$$\|E_{ab}^0\|^{-1} = S + J, \quad (3.4)$$

where S is diagonal and J is antisymmetric:

$$\|S^{ab}\| = \text{diag} \left\{ -G_{11}^{-1}, G_{11}^{-1}, \frac{G_{11}}{G_{11}^2 + H^2}, \frac{G_{11}}{G_{11}^2 + H^2}, G_{\rho\rho}^{-1}, G_{\psi\psi}^{-1}, G_{\alpha\alpha}^{-1}, G_{\beta\beta}^{-1} \right\}, \quad (3.5)$$

$$J^{ab} = \frac{H}{G_{11}^2 + H^2} (\delta_3^a \delta_2^b - \delta_3^b \delta_2^a), \quad (3.6)$$

$$\begin{aligned} G_{11} &= \frac{\rho^2 + L_0^2}{R^2}; & G_{\rho\rho} &= R^2 \frac{(1 + L_0'^2)}{\rho^2 + L_0^2}; & G_{\psi\psi} &= \frac{R^2 \rho^2}{\rho^2 + L_0^2}; \\ G_{\alpha\alpha} &= \cos^2 \psi G_{\psi\psi}; & G_{\beta\beta} &= \sin^2 \psi G_{\psi\psi}. \end{aligned} \quad (3.7)$$

Now it is straightforward to get the effective action. At first order in α' the action for the scalar fluctuations is the first variation of the classical action (2.11) and is satisfied by the classical equations of motion. The equation of motion for the gauge field at first order was considered in section 2 for the computation of the $C_{(6)}$ potential induced by the B -field. Therefore we focus on the second order contribution from the DBI action.

After integrating by parts and taking advantage of the Bianchi identities for the gauge field, we end up with the following terms. For χ :

$$\mathcal{L}_\chi \propto \frac{1}{2} \sqrt{-E^0} \frac{R^2}{\rho^2 + L_0^2} \frac{S^{ab}}{1 + L_0'^2} \partial_a \chi \partial_b \chi + \left[\partial_{L_0}^2 \sqrt{-E^0} - \partial_\rho \left(\partial_{L_0} \sqrt{-E^0} \frac{L_0'}{1 + L_0'^2} \right) \right] \frac{1}{2} \chi^2, \quad (3.8)$$

and for F :

$$\mathcal{L}_F \propto \frac{1}{4} \sqrt{-E^0} S^{aa'} S^{bb'} F_{ab} F_{a'b'} , \quad (3.9)$$

and the mixed χ - F terms:

$$\mathcal{L}_{F\chi} \propto \frac{\sin 2\psi}{2} f \chi F_{23} , \quad (3.10)$$

and for Φ :

$$\mathcal{L}_\Phi \propto \frac{1}{2} \sqrt{-E^0} \frac{R^2 L_0^2}{\rho^2 + L_0^2} S^{ab} \partial_a \Phi \partial_b \Phi , \quad (3.11)$$

where the function f in (3.10) is given by:

$$f(\rho) = \partial_\rho \left(g(\rho) \frac{L_0'}{1 + L_0'^2} J^{23} \right) + J^{32} \partial_{L_0} g(\rho) + 2g(\rho) J^{23} S^{22} \partial_{L_0} G_{11} , \quad (3.12)$$

$$\text{with } g(\rho) = \frac{\sqrt{-E^0}}{\sin \psi \cos \psi} = \rho^3 \sqrt{1 + L_0'^2} \sqrt{1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}} .$$

As can be seen from equation (3.10) the A_2, A_3 components of the gauge field couple to the scalar field χ via the function f . Note that since for $\rho \rightarrow \infty$ and $L \rightarrow \infty$, we see that $J^{23} \rightarrow 0$, the mixing of the scalar and vector field decouples asymptotically. In order to proceed with the analysis we need to take into account the contribution from the Wess-Zumino part of the action. The relevant terms to second order in α' are [9]:

$$S_{WZ} = \frac{(2\pi\alpha')^2}{2} \mu_7 \int F_{(2)} \wedge F_{(2)} \wedge C_{(4)} + (2\pi\alpha') \mu_7 \int F_{(2)} \wedge B_{(2)} \wedge \tilde{P}[C_{(4)}] , \quad (3.13)$$

where $C_{(4)}$ is the background R-R potential given in equation (2.1) and $\tilde{C}_{(4)}$ is the pull back of its magnetic dual. One can show that:

$$\tilde{C}_4 = \frac{R^4}{g_s} \frac{2\rho^2 + L^2}{(\rho^2 + L^2)^2} L^2 \sin \psi \cos \psi d\psi \wedge d\alpha \wedge d\beta \wedge d\phi . \quad (3.14)$$

Writing $\phi = 2\pi\alpha'\Phi$ we write for the pull back $P[\tilde{C}_{(4)}]$:

$$P[\tilde{C}_{(4)}] = -\frac{2\pi\alpha'}{g_s} \frac{\sin 2\psi}{2} K(\rho) \partial_a \Phi d\psi \wedge d\alpha \wedge d\beta \wedge dx^a , \quad (3.15)$$

where we have defined:

$$K(\rho) = -R^4 L_0^2 \frac{2\rho^2 + L_0^2}{(\rho^2 + L_0^2)^2} \quad (3.16)$$

Now note that the B -field has components only along x^2 and x^3 , therefore dx^a in equation (3.15) can be only $d\rho, dx^0$ or dx^1 . This will determine the components of the gauge field which can mix with Φ , However after integrating by parts and using the Bianchi identities one can get the following simple expression for the mixing term:

$$-(2\pi\alpha')^2 \frac{\mu_7}{g_s} \int d^8 \xi \frac{\sin 2\psi}{2} H \partial_\rho K \Phi F_{01} , \quad (3.17)$$

resulting in the following contribution to the complete lagrangian:

$$\mathcal{L}_{F\Phi} \propto \frac{\sin 2\psi}{2} H \partial_\rho K \Phi F_{01} . \quad (3.18)$$

Note that this means that only the A_0 and A_1 components of the gauge field couple to the scalar field Φ . Next the contribution from the first term in (3.13) is given by:

$$(2\pi\alpha')^2 \frac{\mu_7}{g_s} \int d^8\xi \frac{(\rho^2 + L_0^2)^2}{8R^4} F_{ab} F_{cd} \epsilon^{abcd} , \quad (3.19)$$

where the indices take values along the $\rho, \psi, \alpha, \beta$ directions of the world volume. This will contribute to the equation of motion for A_ρ, A_ψ, A_α and A_β , which do not couple to the scalar fluctuations. In this paper we will be interested in analyzing the spectrum of the scalar modes, therefore we will not be interested in the components of the gauge field transverse to the D3-branes world volume. However although there are no sources for these components from the scalar fluctuations, they still couple to the components along the D3-branes as a result setting them to zero will impose constraints on the $A_0 \dots A_3$. Indeed from the equation of motion for the gauge field along the transverse direction one gets:

$$\sum_{a=0}^3 S^{aa} \partial_b \partial_a A_a = 0, \quad b = \rho, \psi, \alpha, \beta , \quad (3.20)$$

(Here, no summation on repeated indices is intended.) However the non-zero B -field explicitly breaks the Lorentz symmetry along the D3-branes' world volume. In particular we have:

$$S^{00} = -S^{11}, \quad S^{22} = S^{33} \neq S^{11}, \quad (3.21)$$

which suggests that we should impose:

$$-\partial_0 A_0 + \partial_1 A_1 = 0, \quad \partial_2 A_2 + \partial_3 A_3 = 0 . \quad (3.22)$$

We will see that these constraints are consistent with the equations of motion for $A_0 \dots A_3$. Indeed with this constraint the equations of motion for χ, Φ and $A_\mu, \mu = 0 \dots 3$ are, for χ :

$$\begin{aligned} \frac{1 + L_0'^2}{g} \partial_\rho \left(\frac{g \partial_\rho \chi}{(1 + L_0'^2)^2} \right) + \frac{\Delta_{\Omega_3} \chi}{\rho^2} + \frac{R^4}{(\rho^2 + L_0^2)^2} \tilde{\square} \chi + \\ + \frac{1 + L_0'^2}{g} \left(-\partial_\rho \left(\frac{\partial g}{\partial L_0} \frac{L_0'}{1 + L_0'^2} \right) + \frac{\partial^2 g}{\partial L_0^2} \right) \chi + \frac{1 + L_0'^2}{g} f F_{23} = 0, \end{aligned} \quad (3.23)$$

and for Φ :

$$\frac{1}{g} \partial_\rho \left(\frac{g L_0^2 \partial_\rho \Phi}{1 + L_0'^2} \right) + \frac{L_0^2 \Delta_{\Omega_3} \Phi}{\rho^2} + \frac{R^4 L_0^2}{(\rho^2 + L_0^2)^2} \tilde{\square} \Phi - \frac{H \partial_\rho K}{g} F_{01} = 0, \quad (3.24)$$

and finally for A_a :

$$\begin{aligned}
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho A_0}{1 + L_0'^2} \right) + \frac{\Delta_{\Omega_3} A_0}{\rho^2} + \frac{R^4}{(\rho^2 + L_0^2)^2} \tilde{\square} A_0 + \frac{H \partial_\rho K}{g} \partial_1 \Phi = 0, \\
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho A_1}{1 + L_0'^2} \right) + \frac{\Delta_{\Omega_3} A_1}{\rho^2} + \frac{R^4}{(\rho^2 + L_0^2)^2} \tilde{\square} A_1 + \frac{H \partial_\rho K}{g} \partial_0 \Phi = 0, \\
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho A_2}{(1 + L_0'^2) \left(1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}\right)} \right) + \frac{R^4}{(\rho^2 + L_0^2)^2 + R^4 H^2} \tilde{\square} A_2 + \frac{\Delta_{\Omega_3} A_2}{\rho^2 \left(1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}\right)} - \frac{f}{g} \partial_3 \chi = 0, \\
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho A_3}{(1 + L_0'^2) \left(1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}\right)} \right) + \frac{R^4}{(\rho^2 + L_0^2)^2 + R^4 H^2} \tilde{\square} A_3 + \frac{\Delta_{\Omega_3} A_3}{\rho^2 \left(1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}\right)} + \frac{f}{g} \partial_2 \chi = 0.
\end{aligned} \tag{3.25}$$

We have defined:

$$\tilde{\square} = -\partial_0^2 + \partial_1^2 + \frac{\partial_2^2 + \partial_3^2}{1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}}. \tag{3.26}$$

As one can see the spectrum splits into two independent components, namely the vector modes A_0, A_1 couple to the scalar fluctuations along Φ , while the vector modes A_2, A_3 couple to the scalar modes along χ . However it is possible to further simplify the equations of motion for the gauge field. Focusing on the equations of motion for A_0 and A_1 in equation (3.25), it is possible to rewrite them as:

$$\begin{aligned}
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho F_{01}}{1 + L_0'^2} \right) + \frac{\Delta_{\Omega_3} F_{01}}{\rho^2} + \frac{R^4}{(\rho^2 + L_0^2)^2} \tilde{\square} F_{01} - \frac{H \partial_\rho K}{g} (-\partial_0^2 + \partial_1^2) \Phi = 0 \\
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho (-\partial_0 A_0 + \partial_1 A_1)}{1 + L_0'^2} \right) + \frac{\Delta_{\Omega_3} (-\partial_0 A_0 + \partial_1 A_1)}{\rho^2} + \frac{R^4}{(\rho^2 + L_0^2)^2} \tilde{\square} (-\partial_0 A_0 + \partial_1 A_1) = 0.
\end{aligned} \tag{3.27}$$

Note that the first constraint in (3.22) trivially satisfies the second equation in (3.27). In this way we are left with the first equation in (3.27). Similarly one can show that using the second constraint in (3.22) the equations of motion in (3.25) for A_2 and A_3 boil down to a single equation for F_{23} :

$$\begin{aligned}
& \frac{1}{g} \partial_\rho \left(\frac{g \partial_\rho F_{23}}{(1 + L_0'^2) \left(1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}\right)} \right) + \frac{R^4}{(\rho^2 + L_0^2)^2 + R^4 H^2} \tilde{\square} F_{23} \\
& \quad + \frac{\Delta_{\Omega_3} F_{23}}{\rho^2 \left(1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}\right)} + \frac{f}{g} (\partial_2^2 + \partial_3^2) \chi = 0.
\end{aligned} \tag{3.28}$$

Now let us proceed with a study of the fluctuations along Φ .

3.2 Fluctuations along Φ

To proceed, we have to take into account the F_{01} component of the gauge field strength and solve the coupled equations of motion. Since the classical solution for the embedding of the D -brane is known only numerically we have to rely again on numerics to study the meson spectrum. However if we look at equation (2.18) we can see that the terms responsible for the non-trivial parts of the equation of motion are of order H^2 . On the other hand, the mixing of the scalar and vector modes due to the term (3.18) appear at first order in H . Therefore it is possible to extract some non-trivial properties of the meson spectrum even at linear order in H and as it turns out, we can observe a Zeeman-like effect: A splitting of states that is proportional to the magnitude of the magnetic field. To describe this, let us study the approximation of weak magnetic field.

3.2.1 Weak magnetic field

To first order in H the classical solution for the D7-brane profile is given by:

$$L_0 = m + O(H^2), \quad (3.29)$$

where m is the asymptotic separation of the D3 and D7-branes and corresponds to the bare quark mass. In this approximation the expressions for $g(\rho)$ and $\partial_\rho K(\rho)$, become:

$$g(\rho) = \rho^3, \quad \partial_\rho K(\rho) = \frac{4m^2 R^4 \rho^3}{(\rho^2 + m^2)^3},$$

and the equations of motion for Φ and F_{01} , equations (3.24) and (3.27), simplify to:

$$\begin{aligned} \frac{1}{\rho^3} (\rho^3 m^2 \partial_\rho \Phi) + \frac{m^2 \Delta_{\Omega_3}}{\rho^2} \Phi + \frac{m^2 R^4}{(\rho^2 + m^2)^2} \square \Phi - 4H \frac{m^2 R^4}{(\rho^2 + m^2)^3} F_{01} &= 0, \quad (3.30) \\ \text{and } \frac{1}{\rho^3} \partial_\rho (\rho^3 \partial_\rho F_{01}) + \frac{\Delta_{\Omega_3} F_{01}}{\rho^2} + \frac{R^4}{(\rho^2 + m^2)^2} \square F_{01} - 4H \frac{m^2 R^4}{(\rho^2 + m^2)^3} \mathcal{P}^2 \Phi &= 0, \end{aligned}$$

$$\text{where } \square = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2, \quad \mathcal{P}^2 = -\partial_0^2 + \partial_1^2.$$

This system has become similar to the system studied in ref. [9] and in order to decouple it we can define the fields:

$$\phi_\pm = F_{01} \pm m \mathcal{P} \Phi, \quad (3.31)$$

where $\mathcal{P} = \sqrt{-\partial_0^2 + \partial_1^2}$. The resulting equations of motion are:

$$\frac{1}{\rho^3} \partial_\rho (\rho^3 \partial_\rho \phi_\pm) + \frac{\Delta_{\Omega_3}}{\rho^2} \phi_\pm + \frac{R^4}{(\rho^2 + m^2)^2} \square \phi_\pm \mp H \frac{4R^4 m}{(\rho^2 + m^2)^3} \mathcal{P} \phi_\pm = 0. \quad (3.32)$$

Note that \mathcal{P}^2 is the Casimir operator in the (x_0, x_1) plane only, while \square is the Casimir operator along the D3-branes' world volume. If we consider a plane wave $e^{ix.k}$ then we can define:

$$\square e^{ix.k} = M^2 e^{ix.k}, \quad \mathcal{P}^2 e^{ix.k} = M_{01}^2 e^{ix.k}, \quad (3.33)$$

and we have the relation:

$$M^2 = M_{01}^2 - k_2^2 - k_3^2. \quad (3.34)$$

The corresponding spectrum of M^2 is continuous in k_2, k_3 . However, if we restrict ourselves to motion in the (x_0, x_1) -plane the spectrum is discrete. Indeed let us consider the ansatz:

$$\phi_{\pm} = \eta_{\pm}(\rho)e^{-ix_0k_0 + ik_1x_1} . \quad (3.35)$$

Then we can write:

$$\frac{1}{\rho^3} \partial_{\rho}(\rho^3 \partial_{\rho} \eta_{\pm}) + \frac{R^4}{(\rho^2 + m^2)^2} M_{\pm}^2 \eta_{\pm} \mp H \frac{4R^4 m}{(\rho^2 + m^2)^3} M_{\pm} \eta_{\pm} = 0, \quad (3.36)$$

$$M_{\pm} \equiv M_{01\pm} .$$

Let us analyze equation (3.36). It is convenient to introduce:

$$y = -\frac{\rho^2}{m^2}; \quad \bar{M}_{\pm} = \frac{R^2}{m} M_{\pm}; \quad P_{\pm}(y) = (1-y)^{\alpha_{\pm}} \eta_{\pm}; \quad (3.37)$$

$$2\alpha_{\pm} = 1 + \sqrt{1 + \bar{M}_{\pm}^2}; \quad \epsilon = H \frac{R^2}{m^2} .$$

With this change of variables equation (3.36) is equivalent to:

$$y(1-y)P_{\pm}'' + 2(1 - (1 - \alpha_{\pm})y)P_{\pm}' - \alpha_{\pm}(\alpha_{\pm-1})P_{\pm} \pm \epsilon \frac{\bar{M}_{\pm}}{(1-y)^2} P_{\pm} = 0 . \quad (3.38)$$

Next we can expand:

$$P_{\pm} = P_0 \pm \epsilon P_1 + O(\epsilon^2); \quad \alpha_{\pm} = \alpha_0 \pm \epsilon \alpha_1 + O(\epsilon^2); \quad (3.39)$$

$$\bar{M}_{\pm} = \bar{M}_0 \pm \epsilon \alpha_1 \frac{(4\alpha_0 + 2)}{\bar{M}_0} + O(\epsilon^2); \quad \bar{M}_0 = 2\sqrt{\alpha_0(\alpha_0 + 1)} .$$

leading to the following equations for P_0 and P_1 :

$$y(1-y)P_0'' + 2(1 - (1 - \alpha_0)y)P_0' - \alpha_0(\alpha_0 - 1)P_0 = 0, \quad (3.40)$$

$$y(1-y)P_1'' + 2(1 - (1 - \alpha_0)y)P_1' - \alpha_0(\alpha_0 - 1)P_1 = \left(\alpha_1(2\alpha_0 - 1) - \frac{\bar{M}_0}{(1-y)^2} \right) P_0 - 2\alpha_1 y P_0' .$$

The first equation in (3.40) is the hypergeometric equation and corresponds to the fluctuations in pure $AdS_5 \times S^5$. It has the regular solution [3]:

$$P_0(y) = F(-\alpha_0, 1 - \alpha_0, 2, y) . \quad (3.41)$$

Furthermore regularity of the solution for $\eta(\rho)$ at infinity requires [3] that α_0 be discrete, and hence the spectrum of \bar{M}_0 :

$$1 - \alpha_0 = -n, \quad n = 0, 1, \dots \quad (3.42)$$

$$\bar{M}_0 = 2\sqrt{(n+1)(n+2)} .$$

The second equation in (3.40) is an inhomogeneous hypergeometric equation. However for the ground state, namely $n = 0$, $P_0 = F(-1, 0, 2, y) = 1$ and one can easily get the solution:

$$P_1(y) = \frac{\bar{M}_0}{6} \ln(1-y) + (6\alpha_1 - \bar{M}_0) \left(\ln(-y) + \frac{1}{y} \right) - \frac{\bar{M}_0}{4(1-y)} . \quad (3.43)$$

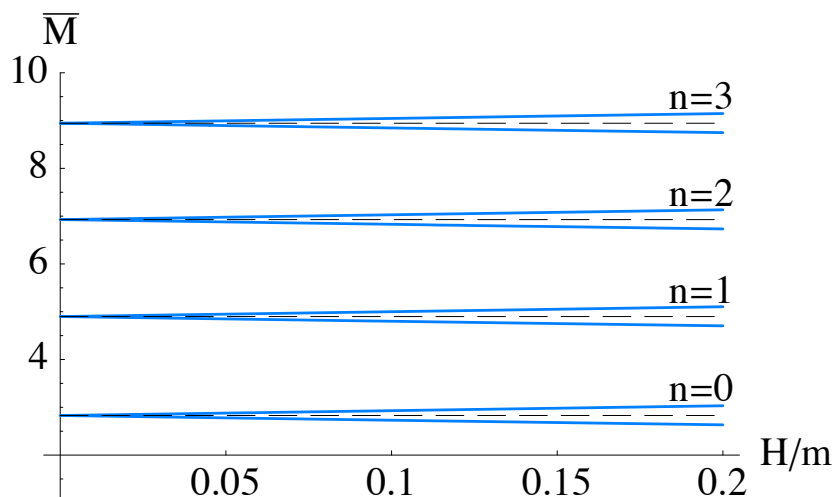


Figure 4: Plot of $\bar{M} = MR^2/m$ vs. H/m for the first three states. The dashed black lines correspond to the spectrum given by equation (3.42)

On the other hand, using the definition of $P_{\pm}(y)$ in (3.37) to first order in ϵ we can write:

$$\eta_{\pm} = \frac{1}{(1-y)^{\alpha_0}} \left(1 \mp \epsilon \frac{\alpha_1}{\alpha_0} \ln(1-y) \right) (1 \pm \epsilon P_1(y)) , \quad (3.44)$$

for the ground state $\alpha_0 = 1$ and we end up with the following expression for η_{\pm} :

$$\eta_{\pm} = \frac{1}{1-y} \pm \epsilon \frac{\bar{M}_0}{4(1-y)^2} \pm \frac{\epsilon}{1-y} (6\alpha_1 - \bar{M}_0) \left(\ln(-y) + \frac{1}{y} - \frac{\ln(1-y)}{6} \right) . \quad (3.45)$$

Now if we require that our solution is regular at $y = 0$ and goes as $1/\rho^2 \propto 1/y$ at infinity, the last term in (3.45) must vanish. Therefore we have:

$$\alpha_1 = \frac{\bar{M}_0}{6} . \quad (3.46)$$

After substituting in (3.39) and (3.37) we end up with the following correction to the ground state:

$$M_{\pm} = M_0 \pm \frac{H}{m} . \quad (3.47)$$

We observe how the introduction of an external magnetic field breaks the degeneracy of the spectrum given by equation (3.42) and results in Zeeman splitting of the energy states, proportional to the magnitude of H . Although equation (3.47) was derived using the ground state it is natural to expect that the same effect takes place for higher excited states. To demonstrate this it is more convenient to employ numerical techniques for solving equation (3.36) and use the methods described in ref. [4] to extract the spectrum. The resulting plot is presented in figure 4. As expected we observe Zeeman splitting of the higher excited states. It is interesting that equation (3.47) describes well not only the ground state, but also the first several excited states.

It turns out that one can easily generalize equation (3.47) to the case of non-zero momentum in the (x_2, x_3) -plane. Indeed if we start from equation (3.32) and proceed with the ansatz:

$$\phi_{\pm} = \tilde{\eta}_{\pm}(\rho)e^{-ix \cdot k}, \quad (3.48)$$

we end up with:

$$\frac{1}{\rho^3} \partial_{\rho}(\rho^3 \partial_{\rho} \tilde{\eta}_{\pm}) + \frac{R^4}{(\rho^2 + m^2)^2} M_{\pm}^2 \tilde{\eta}_{\pm} \mp H \frac{4R^4 m}{(\rho^2 + m^2)^3} M_{01\pm} \tilde{\eta}_{\pm} = 0, \quad (3.49)$$

$$M_{01\pm} = \sqrt{M_{\pm}^2 + k_{23}^2};$$

$$k_{23} \equiv \sqrt{k_2^2 + k_3^2}.$$

After going through the steps described in equations (3.37)–(3.45), equation (3.46) gets modified to:

$$\alpha_1 = \frac{\bar{M}_0}{6} \sqrt{1 + \frac{k_{23}^2}{M_0^2}}. \quad (3.50)$$

Note that validity of the perturbative analysis suggests that α_1 is of the order of α_0 and therefore we can trust the above expression as long as k_{23} is of the order of M_0 . Now it is straightforward to obtain the correction to the spectrum:

$$M_{\pm} = M_0 \pm \frac{H}{m} \sqrt{1 + \frac{k_{23}^2}{M_0^2}}. \quad (3.51)$$

We see that the addition of momentum along the $(x_2 - x_3)$ -plane enhances the splitting of the states. Furthermore the spectrum depends continuously on k_{23} .

3.2.2 Strong magnetic field

For strong magnetic field we have to take into account terms of order H^2 , which means that we no longer have an expression for $L_0(\rho)$ in a closed form and we have to rely on numerical calculations only. Furthermore there is no obvious way to decouple equations (3.24) and (3.27). The numerical analysis of this coupled system of differential equations in the more general case of finite temperature was considered in [8]. The results show that for strong magnetic field the Zeeman splitting described in the previous subsection persist and for sufficiently strong magnetic field the energy levels intersect.

It is interesting to explore the effect of the chiral symmetry breaking on the meson spectrum. It turns out that if we restrict ourselves to fluctuations along the (x^2, x^3) plane, there is no source term in equation (3.25), and we can consistently set F_{01} equal to zero. The resulting spectrum corresponds to the mass in two Euclidean dimensions. Indeed let us consider the ansatz:

$$\Phi = h(\rho)e^{-ik_2 x^2 - ik_3 x^3} Y_l(S^3), \quad (3.52)$$

where $Y_l(S^3)$ are the spherical harmonics on the S^3 sphere satisfying: $\Delta_{\Omega_3} Y_l = -l(l+2)Y_l$. With this set-up the equation of motion for Φ , equation (3.24), reduces to equation for

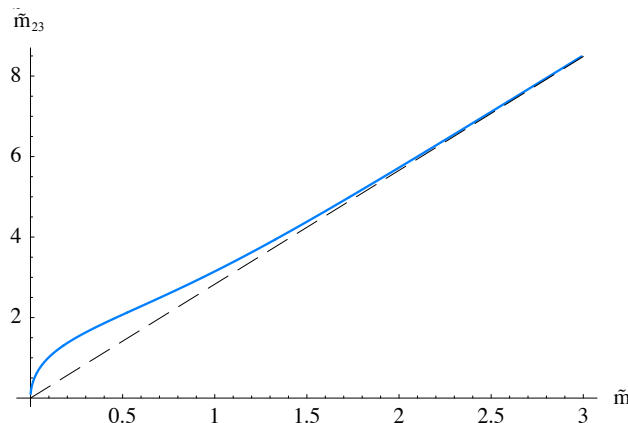


Figure 5: Spectrum of \tilde{m}_{23} vs. \tilde{m} . The dashed line represents the lowest level of the meson spectrum for pure $AdS_5 \times S^5$ space

$h(\rho)$:

$$\frac{1}{g} \partial_\rho \left(\frac{gL_0^2 \partial_\rho h(\rho)}{1 + L_0'^2} \right) - \frac{L_0^2 l(l+2)}{\rho^2} h(\rho) + \frac{R^4 L_0^2 m_{23}^2}{(\rho^2 + L_0^2)^2 + R^4 H^2} h(\rho) = 0, \quad (3.53)$$

where we have defined:

$$m_{23}^2 = -k_2^2 - k_3^2. \quad (3.54)$$

Before we proceed with the numerical analysis of equation (3.53) let us introduce dimensionless variables by performing the transformation (2.20) and defining:

$$\tilde{m}_{23} = \frac{R}{\sqrt{H}} m_{23}. \quad (3.55)$$

The resulting equation is:

$$\begin{aligned} \frac{\tilde{\rho}^2 + \tilde{L}^2}{\tilde{\rho}^3 \sqrt{1 + \tilde{L}'^2} (1 + (\tilde{\rho}^2 + \tilde{L}^2)^2)^{1/2}} \partial_{\tilde{\rho}} \left(\tilde{\rho}^3 \left(1 + \frac{1}{(\tilde{\rho}^2 + \tilde{L}^2)^2} \right)^{1/2} \frac{\tilde{L}^2}{\sqrt{1 + \tilde{L}'^2}} \partial_{\tilde{\rho}} h(\tilde{\rho}) \right) + \\ - \frac{\tilde{L}^2}{\tilde{\rho}^2} l(l+2) h(\tilde{\rho}) + \frac{\tilde{L}^2 \tilde{m}_{23}^2}{(\tilde{\rho}^2 + \tilde{L}^2)^2 + 1} h(\tilde{\rho}) = 0. \end{aligned} \quad (3.56)$$

In order to study the spectrum we look for normalizable solutions which have asymptotic behavior $h(\tilde{\rho}) \propto 1/\tilde{\rho}^2$ for large $\tilde{\rho}$ and satisfy the following boundary conditions at $\tilde{\rho} = 0$:

$$h'(0) = 0; \quad h(0) = 1. \quad (3.57)$$

Let us consider first the lowest level of the spectrum. The spectrum that we get as a function of the bare quark mass is plotted in figure 5.

For large \tilde{m} the spectrum asymptotes (the dashed line in figure 5) to the one for pure $AdS_5 \times S^5$ space obtained by ref. [3]

$$M_0 = \frac{2m}{R^2} \sqrt{(n+l+1)(n+l+2)}, \quad (3.58)$$

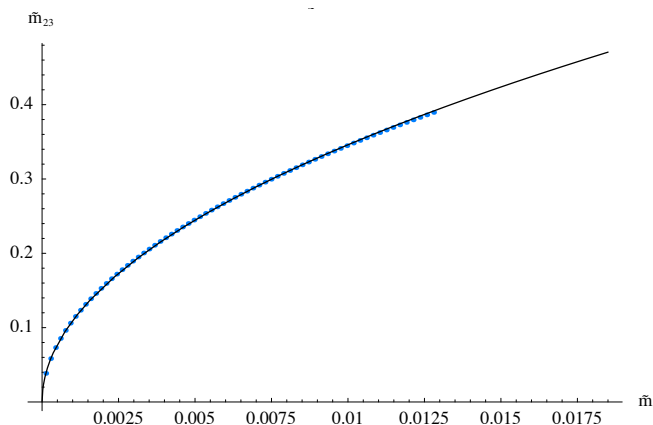


Figure 6: Enlargement of part of the spectrum of \tilde{m}_{23} vs \tilde{m} from figure 5. The black solid curve shows the $\propto \sqrt{\tilde{m}}$ fit.

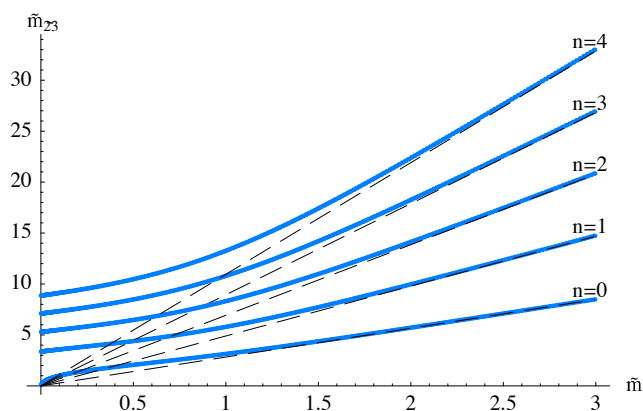


Figure 7: Spectrum of \tilde{m}_{23} vs \tilde{m} for $n = 0 \dots 4$. The dashed lines represent the spectrum for $AdS_5 \times S^5$ space.

with the substitution $n = 0, l = 0$, to obtain our case. Therefore we are describing the lowest possible state of the meson spectrum. In figure 6 we have zoomed in the area near the origin of the $(\tilde{m}, \tilde{m}_{23})$ -plane, one can see that for small values of $\tilde{m} = 2\pi\alpha' m_q / R\sqrt{H}$ we observe $\propto \sqrt{m_q}$ dependence of the ground state on the bare quark mass m_q , which is to be expected since the chiral symmetry associated with the spinor representation of $SO(2)$ is spontaneously broken [41].

It is interesting to look for modes corresponding to higher excited states (non-zero n). In figure 7 we have presented a plot of some of these. Again, the dashed line correspond to the pure $AdS_5 \times S^5$ spectrum given by (3.58) for $l = 0$. For small values of \tilde{m} one can observe the qualitative difference of the behavior of the spectrum corresponding to the $n = 0$ state from that of the higher excited states. Indeed as $\tilde{m} \rightarrow 0$ the $n = 0$ states follow the $\sqrt{\tilde{m}}$ behavior plotted in figure 6, while the excited states tend to some finite

values at zero bare quark mass. The $n = 0$ states merge into the Goldstone boson of the spontaneously broken chiral symmetry.

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